

Duality of W^* -correspondences and applications

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1 Introduction

In [22] Pimsner used C^* -correspondences to construct and study a rich class of C^* -algebras. In our work [11] we introduced and studied a class of nonselfadjoint operator algebras, called tensor algebras, that are constructed from C^* -correspondences and are subalgebras of Pimsner's C^* -algebras. Later, in [16], we studied the Hardy algebras (which are the ultraweak closures of tensor algebras associated with correspondences over von Neumann algebras). Together they form a rich class of operator algebras containing a large variety of algebras (such as analytic crossed products [9, 21], noncommutative disc algebras [25], free semigroup algebras [5], quiver or free semigroupoid algebras [12, 7] and others). The definition of a C^* -correspondence and the construction of the tensor algebra associated with it will be presented in Section 2.

The simplest example of a tensor algebra is the classical disc algebra $A(\mathbb{D})$ (associated with the correspondence \mathbb{C} over the algebra \mathbb{C}). The study of its representation theory amounts to the study of contraction operators (up to unitary equivalence). It turns out that many ingredients of that theory can be

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generalized to the study of the representation theory of general tensor or Hardy algebras. In this paper we shall deal mostly with the Hardy algebras. The simplest one is the classical $H^\infty(\mathbb{T})$.

Model theory was originally formulated by Sz-Nagy and Foias and others (see [19]) to study contraction operators on Hilbert space. In Section 4 we describe how one can define canonical models for certain representations of the Hardy algebra. This is mainly an exposition of some of the results in [17]. It generalizes some of the classical results as well as the results of Popescu [24] for row contractions. Our goal is to establish a bijective correspondence between the completely non coisometric representations of the Hardy algebra and characteristic operator functions.

In the classical theory the characteristic operator function defining the model is a Schur multiplier; i.e. an H^∞ function defined on \mathbb{D} taking values in $B(\mathcal{E}_1, \mathcal{E}_2)$ (for a pair of Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$). When $H^\infty(\mathbb{D})$ is replaced by a general Hardy algebra (denoted $H^\infty(E)$, where E is a W^* -correspondence over a von Neumann algebra M) the “Schur multipliers” are elements of another Hardy algebra. That Hardy algebra is associated with a correspondence that is “dual” to E in a sense that we shall make precise in Definition 3.6.

The concept of the dual of a W^* -correspondence may be traced back at least to Rieffel’s pioneering work on Morita equivalence for C^* - and W^* -algebras [27], although the terminology did not appear until [16]. Our thinking about the notion was stimulated by the work of Arveson [2] where he associated a Hilbert space (i.e. a correspondence over \mathbb{C}) to an endomorphism α of $B(H)$. Essentially, he showed that this Hilbert space is dual to the correspondence ${}_\alpha B(H)$ associated with the endomorphism. Arveson’s construction was generalized by us [14] to yield a correspondence E_Θ associated to a completely positive map Θ on a general von Neumann algebra M . The correspondence E_Θ is the dual of the correspondence defined by Paschke in [20] as a generalization of the *GNS* construction associated with a state on a C^* -algebra and studied extensively in [23], [1], [10] and elsewhere. Nowadays it is known as the *GNS-correspondence* associated to Θ (see [4] especially). In fact, it can be shown that the techniques of [14] that yield endomorphic dilations for semigroups of completely positive maps are, in a sense that may be made precise, dual to the techniques used by Bhat and Skieda in [4] to achieve their dilation result. A discussion of the relation between [14] and [4] appears in [28] and will be developed further in [18]. Duality of correspondences has also proved useful in the analysis of “curvature” for completely positive maps [15] and in our work on the Hardy algebra of a W^* -correspondence [16]. It is in study of Hardy algebras where duality first appeared in our thinking even before the appearance of [14] (although [16] was completed much later).

In Section 3 we define and discuss duality for W^* -correspondences and present the duality theorem (Theorem 3.6). In fact, we shall generalize here some results (and definitions) of [16] to the context of W^* -correspondences from one von Neumann algebra M to another, N . (In [16] the main results were proved under the assumption that $M = N$). This will allow us to present and sketch the proof of Theorem 3.11 using duality to give necessary and suf-

ficient conditions for two W^* -correspondences to be Morita equivalent (in the sense of [13]). Full details will be developed in [18].

In the next section we introduce the notation and constructions used throughout the paper.

2 Correspondences and operator algebras

We start by introducing the basic definitions and constructions. We shall follow Lance [8] for the general theory of Hilbert C^* -modules that we shall use. Let A be a C^* -algebra and E be a right module over A endowed with a bi-additive map $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$ (referred to as an A -valued inner product) such that, for $\xi, \eta \in E$ and $a \in A$, $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$, $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$, and $\langle \xi, \xi \rangle \geq 0$, with $\langle \xi, \xi \rangle = 0$ only when $\xi = 0$. Also, E is assumed to be complete in the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$. We write $\mathcal{L}(E)$ for the space of continuous, adjointable, A -module maps on E . It is known to be a C^* -algebra. If M is a von Neumann algebra and if E is a Hilbert C^* -module over M , then E is said to be *self-dual* in case every continuous M -module map from E to M is given by an inner product with an element of E . Let A and B be C^* -algebras. A C^* -correspondence from A to B is a Hilbert C^* -module E over B endowed with a structure of a left module over A via a nondegenerate $*$ -homomorphism $\varphi : A \rightarrow \mathcal{L}(E)$.

When dealing with a specific C^* -correspondence, E , from a C^* -algebra A to a C^* -algebra B , it will be convenient to suppress the φ in formulas involving the left action and simply write $a\xi$ or $a \cdot \xi$ for $\varphi(a)\xi$. This should cause no confusion in context.

C^* -correspondences should be viewed as generalized C^* -homomorphisms. Indeed, the collection of C^* -algebras together with (isomorphism classes of) C^* -correspondences is a category that contains (contravariantly) the category of C^* -algebras and (conjugacy classes of) C^* -homomorphisms. Of course, for this to make sense, one has to have a notion of composition of correspondences and a precise notion of isomorphism. The notion of isomorphism is the obvious one: a bijective, bimodule map that preserves inner products. Composition is “tensoring”: If E is a C^* -correspondence from A to B and if F is a correspondence from B to C , then the balanced tensor product, $E \otimes_B F$ is an A, C -bimodule that carries the inner product defined by the formula

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{E \otimes_B F} := \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle_E) \eta_2 \rangle_F$$

The Hausdorff completion of this bimodule is again denoted by $E \otimes_B F$ and is called the *composition* of E and F . At the level of correspondences, composition is not associative. However, if we pass to isomorphism classes, it is. That is, we only have an isomorphism $(E \otimes F) \otimes G \simeq E \otimes (F \otimes G)$. It is worthwhile to emphasize here that while it often is safe to ignore the distinction between correspondences and their isomorphism classes, at times the distinction is of critical importance.

In this paper we deal mostly with correspondences over von Neumann algebras that satisfy some natural additional properties as indicated in the following

definitions.

Definition 2.1 *Let N be von Neumann algebra and let E be a Hilbert C^* -module over N . Then E is called a Hilbert W^* -module over N in case E is self-dual.*

Remark 2.2 *When E is a W^* -module over a W^* -algebra N , then E carries a natural topology making E a dual space. This was proved by Paschke in [20, Proposition 3.8], where he shows that E may be identified with a weak- $*$ closed subspace of the dual of the projective tensor product of the complex conjugate of E with the pre-dual of N . The weak- $*$ topology on E is called the σ -topology. It follows easily that $\mathcal{L}(E)$ carries a natural topology making $\mathcal{L}(E)$ a dual space [20, Remark 3.9 and Proposition 3.10]. Thus, since $\mathcal{L}(E)$ is already a C^* -algebra, we see that it is an abstract W^* -algebra. For more details about the σ -topology see [3].*

Definition 2.3 *If M and N are two von Neumann algebras, then an M - N -bimodule E is called a W^* -correspondence from M to N in case E is a self-dual C^* -correspondence from M to N such that the $*$ -homomorphism $\varphi : M \rightarrow \mathcal{L}(E)$ giving the left module structure on E is normal. (Note: This makes sense by virtue of the preceding remark.)*

If $M = N$ we shall say that E is a W^ -correspondence over M .*

Remark 2.4 *An isomorphism of a W^* -correspondence E_1 from M_1 to N_1 and a W^* -correspondence E_2 from M_2 to N_2 is a triple (σ, Ψ, τ) where $\sigma : M_1 \rightarrow M_2$ and $\tau : N_1 \rightarrow N_2$ are isomorphisms of von Neumann algebras, $\Psi : E_1 \rightarrow E_2$ is a vector space isomorphism preserving the σ -topology and for $\xi, \eta \in E_1$ and $a \in M_1, b \in N_1$, we have $\Psi(a\xi b) = \sigma(a)\Psi(\xi)\tau(b)$ and $\langle \Psi(\xi), \Psi(\eta) \rangle = \tau(\langle \xi, \eta \rangle)$. When dealing with correspondences over M and over N (i.e. when $M_i = N_i$, $i = 1, 2$), we shall require that $\sigma = \tau$ (unless we say otherwise).*

It is evident that the composition of two W^* -correspondences is a C^* -correspondence. However, it is not in general a W^* -correspondence. One must form its “self-dual completion”. A few words about this may be helpful. Suppose Z is a Hilbert module over a von Neumann algebra N . Then, as Paschke showed in Theorem 3.2 of [20], Z may be embedded in a W^* -module X over N in such a way that Z is dense in X with respect to the σ -topology. Further, X is unique up to isomorphism and so may be referred to as *the self-dual completion of Z* . Paschke’s proof requires a passage to the Banach space double dual of the von Neumann algebra N . In Proposition 6.10 of [27], Rieffel gives an alternate approach to the notion of the self-dual completion that works as well for correspondences. That is, he shows that if M and N are von Neumann algebras and if Z is a C^* -correspondence from M to N , then there is an essentially unique W^* -correspondence X from M to N that contains Z as a subspace that is dense in the σ -topology. Rieffel’s proof also uses “duality” techniques, but of the kind that we discuss below.

Definition 2.5 *Let M , N and P be three von Neumann algebras, let Y be a W^* -correspondence from M to N and let Z be a W^* -correspondence from N to P . Then the self-dual completion of the C^* -correspondence from M to P , $Y \otimes_N Z$, is called the W^* -tensor product of Y and Z (balanced over N) and is also denoted $Y \otimes_N Z$.*

Since we will be considering only W^* -tensor products of W^* -correspondences in this note, there should be no confusion caused from the potential dual use of the notation $Y \otimes_N Z$.

Note also, in particular, that a W^* -correspondence from a von Neumann algebra N to \mathbb{C} is a Hilbert space H equipped with a (normal) representation of N . If E is a W^* -correspondence from M to N then the W^* -tensor product, $E \otimes_N H$, is a Hilbert space equipped with a normal representation of M . If σ is the representation of N on H we shall also write $E \otimes_\sigma H$ for this tensor product. Observe that given an operator $X \in \mathcal{L}(E)$ and an operator $S \in \sigma(N)'$, the map $\xi \otimes h \mapsto X\xi \otimes Sh$ defines a bounded operator on $E \otimes_\sigma H$ denoted by $X \otimes S$. This is a consequence of Theorem 6.23 in [26].

Observe that if E is a W^* -correspondence over a von Neumann algebra M , then each of the tensor powers of E , is a W^* -correspondence over M and so, too, is the full Fock space $\mathcal{F}(E)$, which is defined to be the direct sum $M \oplus E \oplus E^{\otimes 2} \oplus \cdots$, with its obvious structure as a right Hilbert module over M and left action given by the map φ_∞ , defined by the formula $\varphi_\infty(a) := \text{diag}\{a, \varphi(a), \varphi^{(2)}(a), \varphi^{(3)}(a), \dots\}$, where for all n , $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$, $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in E^{\otimes n}$.

The *tensor algebra* over E , denoted $\mathcal{T}_+(E)$, is defined to be the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\varphi_\infty(M)$ and the *creation operators* T_ξ , $\xi \in E$, defined by the formula $T_\xi \eta = \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$. We refer the reader to [11] for the basic facts about $\mathcal{T}_+(E)$.

Definition 2.6 *Given a W^* -correspondence E over the von Neumann algebra M , the ultraweak closure of the tensor algebra of E , $\mathcal{T}_+(E)$, in $\mathcal{L}(\mathcal{F}(E))$, will be called the Hardy Algebra of E , and will be denoted $H^\infty(E)$.*

Example 2.7 *If $M = E = \mathbb{C}$ then $\mathcal{F}(E)$ can be identified with $H^2(\mathbb{T})$. The tensor algebra then is isomorphic to the disc algebra $A(\mathbb{D})$ and the Hardy algebra is the classical Hardy algebra $H^\infty(\mathbb{T})$.*

Example 2.8 *If $M = \mathbb{C}$ and $E = \mathbb{C}^n$ then $\mathcal{F}(E)$ can be identified with the space $l_2(\mathbb{F}_n^+)$ where \mathbb{F}_n^+ is the free semigroup on n generators. The tensor algebra then is what Popescu referred to as the “non commutative disc algebra” \mathcal{A}_n and the Hardy algebra is its w^* -closure. It was studied by Popescu [25] and by Davidson and Pitts who denoted it by \mathcal{L}_n [5].*

Example 2.9 *Let M be a von Neumann algebra and α be an injective normal $*$ -endomorphism on M . The correspondence E associated with α is equal to M as a vector space. The right action is by multiplication, the M -valued inner product is $\langle a, b \rangle = a^*b$ and the left action is given by α ; i.e. $\varphi(a)b = \alpha(a)b$. We*

write ${}_{\alpha}M$ for E . It is easy to check that $E^{\otimes n}$ is isomorphic to ${}_{\alpha^n}M$. The Hardy algebra in this case will be referred to as the non selfadjoint crossed product of M by α and is related to the algebras studied in [9] and [21].

Example 2.10 If α is the identity endomorphism of M , the correspondence ${}_{\alpha}M$ is called the identity correspondence over M .

Example 2.11 Suppose now that Θ is a normal, contractive, completely positive map on a von Neumann algebra M . Then we can associate with it the correspondence $M \otimes_{\Theta} M$ obtained by defining on the algebraic tensor product $M \otimes M$ the M -valued inner product $\langle a \otimes b, c \otimes d \rangle = b^* \Theta(a^* c) d$ and completing. (The bimodule structure is by left and right multiplications). This correspondence, first defined by Paschke in [20], was used by Popa [23], Mingo [10], Anantharam-Delarouche [1] and others to study the map Θ . (In [4] it is referred to as the GNS-module). If Θ is an endomorphism this correspondence is the one described in Example 2.9; that is, if Θ is an endomorphism of M , then $M \otimes_{\Theta} M = {}_{\Theta}M$.

In most respects, the representation theory of $H^{\infty}(E)$ follows the lines of the representation theory of $\mathcal{T}_+(E)$. However, there are some differences that will be important here. To help illuminate these, we need to review some of the basic ideas from [11, 12, 14].

Definition 2.12 Let E be a W^* -correspondence over a von Neumann algebra M . Then:

1. A completely contractive covariant representation of E on a Hilbert space H is a pair (T, σ) , where
 - (a) σ is a normal $*$ -representation of N in $B(H)$.
 - (b) T is a linear, completely contractive map from E to $B(H)$ that is continuous in the σ -topology on E (Remark 2.2) and the ultraweak topology on $B(H)$.
 - (c) T is a bimodule map in the sense that $T(S\xi R) = \sigma(S)T(\xi)\sigma(R)$, $\xi \in E$, and $S, R \in M$.
2. A completely contractive covariant representation (T, σ) of E in $B(H)$ is called isometric in case

$$T(\xi)^* T(\eta) = \sigma(\langle \xi, \eta \rangle) \quad (1)$$

for all $\xi, \eta \in E$.

It should be noted that the operator space structure on E to which Definition 2.12 refers is that which E inherits when viewed as a subspace of its linking algebra. Also, we shall refer to an isometric, completely contractive, covariant representation simply as an isometric covariant representation. There

is no problem with doing this because it is easy to see that if one has a pair (T, σ) satisfying all the conditions of part 1 of Definition 2.12, except possibly the complete contractivity assumption, but which is isometric in the sense of equation (1), then necessarily T is completely contractive. (See [11].)

As we showed in [11, Lemmas 3.4–3.6] and in [16], if a completely contractive covariant representation, (T, σ) , of E in $B(H)$ is given, then it determines a contraction $\tilde{T} : E \otimes_\sigma H \rightarrow H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in E \otimes_\sigma H$. The operator \tilde{T} intertwines the representation σ on H and the induced representation $\sigma^E := \varphi(\cdot) \otimes I_H$ on $E \otimes_\sigma H$; i.e.

$$\tilde{T}(\varphi(\cdot) \otimes I) = \sigma(\cdot)\tilde{T}. \quad (2)$$

In fact we have the following lemma from [16, Lemma 2.16].

Lemma 2.13 *The map $(T, \sigma) \rightarrow \tilde{T}$ is a bijection between all completely contractive covariant representations (T, σ) of E on the Hilbert space H and contractive operators $\tilde{T} : E \otimes_\sigma H \rightarrow H$ that satisfy equation (2). Given such a \tilde{T} satisfying this equation, T , defined by the formula $T(\xi)h := \tilde{T}(\xi \otimes h)$, together with σ is a completely contractive covariant representation of E on H . Further, (T, σ) is isometric if and only if \tilde{T} is an isometry.*

Associated with (T, σ) we also have maps $\tilde{T}_n : E^{\otimes n} \otimes H \rightarrow H$ defined by $\tilde{T}_n(\xi_1 \otimes \xi_2 \cdots \otimes \xi_n \otimes h) = T(\xi_1)T(\xi_2) \cdots T(\xi_n)h$.

In the next theorem we give a complete description of the representations of the tensor algebra. The main ingredient in the proof of this theorem is the construction, to a given completely contractive covariant representation (T, σ) of E , of an isometric representation (V, π) that dilates it and is the minimal isometric dilation. Since we shall later need the notation set in this construction, we briefly describe it. (For full details see [11]).

Given (T, σ) on H we set $\Delta = (I - \tilde{T}^*\tilde{T})^{1/2}$ (in $B(E \otimes_\sigma H)$), $\Delta_* = (I - \tilde{T}\tilde{T}^*)^{1/2}$ (in $B(H)$), $\mathcal{D} = \Delta(E \otimes_\sigma H)$ and $\mathcal{D}_* = \Delta_*(H)$. Also let $L_\xi : H \rightarrow E \otimes_\sigma H$ be the map $L_\xi h = \xi \otimes h$ and $D(\xi) = \Delta \circ L_\xi : H \rightarrow E \otimes_\sigma H$. Note that $T(\xi) = \tilde{T} \circ L_\xi$.

The representation space K of (V, π) is

$$K = H \oplus \mathcal{D} \oplus (E \otimes_{\sigma_1} \mathcal{D}) \oplus (E^{\otimes 2} \otimes_{\sigma_1} \mathcal{D}) \oplus \dots$$

where $\sigma_1(a)$ is the restriction to \mathcal{D} of $\varphi(a) \otimes I_H$. The representation π can be defined by $\pi = \text{diag}(\sigma, \sigma_1, \sigma_2, \dots)$ where $\sigma_{k+1}(a) = \varphi_k(a) \otimes I_{\mathcal{D}}$. The map $V : E \rightarrow B(K)$ is defined by

$$V(\xi) = \begin{pmatrix} T(\xi) & 0 & 0 & \cdots \\ D(\xi) & 0 & 0 & \cdots \\ 0 & L_\xi & 0 & \\ 0 & 0 & L_\xi & \\ & & & \ddots \end{pmatrix} \quad (3)$$

where L_ξ here is the obvious map from $E^{\otimes m} \otimes \mathcal{D}$ to $E^{\otimes(m+1)} \otimes \mathcal{D}$.

It turns out that (V, π) is a covariant isometric representation of E , it dilates (T, σ) in the sense that, for $\xi \in E$ and $a \in M$, $V(\xi)^*$ and $\pi(a)$ leave H invariant and their restrictions to H are equal to $T(\xi)^*$ and $\sigma(a)$ respectively. It is also a minimal dilation (in an obvious sense) and can be shown to be *the unique* (up to unitary equivalence) minimal isometric dilation of (T, σ) .

We now write \mathcal{Q}_0 for the space of all vectors perpendicular to every vector of the form $V(\xi)k$, $\xi \in E$, $k \in K$. Note that \mathcal{Q}_0 is $\pi(M)$ -invariant and, for every $\xi_1, \xi_2, \dots, \xi_m \in E$ and k_1, k_2 in \mathcal{Q}_0 , $\langle V(\xi_1)V(\xi_2) \cdots V(\xi_m)k_1, k_2 \rangle = 0$. Such a subspace of K is said to be *wandering*. It is easy to see that \mathcal{D} is also a wandering subspace. Whenever $\mathcal{M} \subseteq K$ is a wandering subspace, there is a unitary operator, denoted $W(\mathcal{M})$, from $\mathcal{F}(E) \otimes_{\pi|_{\mathcal{M}}} \mathcal{M}$ onto the $V(E)$ -invariant subspace of K generated by \mathcal{M} (denoted $L_\infty(\mathcal{M})$). We also note that there is an isometry u from \mathcal{Q}_0 onto \mathcal{D}_* that commutes with $\pi(a)$ for $a \in M$. (See [17]).

Theorem 2.14 *Let E be a W^* -correspondence over a von Neumann algebra M . To every completely contractive covariant representation, (T, σ) , of E there is a unique completely contractive representation ρ of the tensor algebra $\mathcal{T}_+(E)$ that satisfies*

$$\rho(T_\xi) = T(\xi) \quad \xi \in E$$

and

$$\rho(\varphi_\infty(a)) = \sigma(a) \quad a \in M.$$

The map $(T, \sigma) \mapsto \rho$ is a bijection between the set of all completely contractive covariant representations of E and all completely contractive (algebra) representations of $\mathcal{T}_+(E)$ whose restrictions to $\varphi_\infty(M)$ are continuous with respect to the ultraweak topology on $\mathcal{L}(\mathcal{F}(E))$.

Definition 2.15 *If (T, σ) is a completely contractive covariant representation of a W^* -correspondence E over a von Neumann algebra M , we call the representation ρ of $\mathcal{T}_+(E)$ described in Theorem 2.14 the integrated form of (T, σ) and write $\rho = \sigma \times T$.*

Remark 2.16 *One of the principal difficulties one faces in dealing with $\mathcal{T}_+(E)$ and $H^\infty(E)$ is to decide when the integrated form, $\sigma \times T$, of a completely contractive covariant representation (T, σ) extends from $\mathcal{T}_+(E)$ to $H^\infty(E)$. This problem arises already in the simplest situation, viz. when $M = \mathbb{C} = E$. In this setting, T is given by a single contraction operator on a Hilbert space, $\mathcal{T}_+(E)$ “is” the disc algebra and $H^\infty(E)$ “is” the space of bounded analytic functions on the disc. The representation $\sigma \times T$ extends from the disc algebra to $H^\infty(E)$ precisely when there is no singular part to the spectral measure of the minimal unitary dilation of T . We are not aware of a comparable result in our general context but we have some sufficient conditions. One of them is given in the following lemma. It is not necessary in general.*

Lemma 2.17 [16, Corollary 2.14] *If $\|\tilde{T}\| < 1$ then $\sigma \times T$ extends to a σ -weakly continuous representation of $H^\infty(E)$.*

3 Duality of W^* -correspondences

In this section we discuss the concept of duality for W^* -correspondences. As we noted in the introduction, the concept arises implicitly in [26]. It was used in [2], again implicitly, to construct the product system associated to an E_0 -semigroup on $B(H)$ and a bit more explicitly in [14] where the construction was extended to E_0 -semigroups on a general von Neumann algebra.

Definition 3.1 *Let E be a W^* -correspondence from M to N . Let $\sigma : M \rightarrow B(H)$ and $\tau : N \rightarrow B(K)$ be normal representations of the von Neumann algebras M and N . Then the τ - σ -dual of E , denoted $E^{\tau, \sigma}$, is defined to be*

$$\{\eta \in B(H, E \otimes_{\tau} K) \mid \eta\sigma(a) = (\varphi(a) \otimes I)\eta, a \in M\}.$$

An important feature of the dual $E^{\tau, \sigma}$ is that it is a W^* -correspondence - over $\sigma(M)'$ - as the following proposition shows. (cf. [27, Theorem 6.5]).

Proposition 3.2 *With respect to the actions of $\sigma(M)'$ and $\tau(N)'$ and the $\sigma(M)'$ -valued inner product defined as follows, $E^{\tau, \sigma}$ becomes a W^* -correspondence from $\tau(N)'$ to $\sigma(M)'$: For $Y \in \sigma(M)'$, $X \in \tau(N)'$ and $T \in E^{\tau, \sigma}$, $X \cdot T \cdot Y := (I \otimes X)TY$, and for $T, S \in E^{\tau, \sigma}$, $\langle T, S \rangle_{\sigma(M)'} := T^*S$.*

Remark 3.3 *When $M = N$ (i.e. when E is a W^* -correspondence over M) and when $\tau = \sigma$, we write E^{σ} in place of $E^{\sigma, \sigma}$. The importance of this space for us lies in the fact that it is closely related to the representations of E . In fact, the operators in E^{σ} whose norms do not exceed 1 are precisely the adjoints of the operators of the form \tilde{T} for a covariant pair (T, σ) . In particular, every η in the open unit ball of E^{σ} (written $\mathbb{D}(E^{\sigma})$) gives rise to a covariant pair (T, σ) (with $\eta = \tilde{T}^*$) such that $\sigma \times T$ is a representation of $H^{\infty}(E)$. Given $X \in H^{\infty}(E)$ we can apply this representation (associated to η) to it. The resulting operator in $B(H)$ will be denoted by $X(\eta^*)$. In this way, we view every element in the Hardy algebra as a $B(H)$ -valued function on the unit ball $\mathbb{D}(E^{\sigma})$. This point of view is exploited in [16] to deal with interpolation problems.*

Example 3.4 *Suppose $M = E = \mathbb{C}$ and suppose σ is the representation of \mathbb{C} on some Hilbert space H given by scalar multiplication. Then it is easy to check that E^{σ} is isomorphic to $B(H)$. Fix an $X \in H^{\infty}(E)$. As we mentioned above, this Hardy algebra is the classical space $H^{\infty}(\mathbb{T})$ and we can identify X with a function $f \in H^{\infty}(\mathbb{T})$. Given $S \in E^{\sigma} = B(H)$, it is not hard to check that $X(S^*)$, as defined above, is the operator that arises through the H^{∞} -functional calculus, $f(S^*)$.*

Example 3.5 *If Θ is a contractive, normal, completely positive map on a von Neumann algebra M and if $E = M \otimes_{\Theta} M$ (see Example 2.11) then, for every faithful representation σ of M on H , the σ -dual is the space of all bounded operators mapping H into the Stinespring space K associated with Θ as a map from M to $B(H)$ that intertwine the representation σ (on H) and the Stinespring*

representation π (on K). This correspondence has proved very useful in the study of completely positive maps. (See [14] and [15]). If $M = B(H)$ this is a Hilbert space and was studied by Arveson [2]. Note also that, if Θ is an endomorphism, then this dual correspondence is the space of all operators on H intertwining σ and $\sigma \circ \Theta$.

We now return to discuss the general case (where M is not necessarily equal to N). The term “dual” that we use is justified by the following theorem, which is proved as Theorem 3.6 in [16] under the assumption that $M = N$. See also [29]. It really is contained in Proposition 6.10 of [27], although the proof given in [16] and outlined below is more elementary and transparent. The full result along with applications to Morita theory (in particular Theorem 3.11) will be proved in [18].

Theorem 3.6 *Let E be a W^* -correspondence from M to N and let σ, τ be faithful, normal representations of M (on H) and N (on K) respectively. If we write ι_1 for the identity representation of $\sigma(M)'$ (on H) and ι_2 for the identity representation of $\tau(N)'$ (on K), then one can form the ι_1 - ι_2 -dual of $E^{\tau, \sigma}$ and we have*

$$(E^{\tau, \sigma})^{\iota_1, \iota_2} \cong E.$$

The isomorphism in the theorem is (σ, Ψ, τ) from E onto $(E^{\tau, \sigma})^{\iota_1, \iota_2}$ where Ψ is defined by the equation

$$\Psi(\xi)^*(\eta \otimes h) = L_\xi^* \eta(h),$$

with $\xi \in E$, $\eta \in E^{\tau, \sigma}$ and $h \in H$. Note that η is a map from H to $E \otimes K$ so that $L_\xi^* \eta(h)$ lies in K and the equation above defines a map from $E^{\tau, \sigma} \otimes H$ to K whose adjoint can be shown to have the intertwining property required from an element of $(E^{\tau, \sigma})^{\iota_1, \iota_2}$. In order to prove that Ψ is onto one uses the following lemma. It was proved in [16, Lemma 3.5] for the case $\sigma = \tau$. The proof in the general case requires only a minor adjustment.

Lemma 3.7 *When E is as above and σ and τ are faithful representations of M and N respectively, we have*

$$\bigvee \{X(H) : X \in E^{\tau, \sigma}\} = E \otimes_\tau K.$$

The following two lemmas show that the operation of taking duals “behaves nicely” with respect to direct sums and tensor products.

Lemma 3.8 *Given W^* -correspondences E_1 and E_2 from M to N and faithful representations σ (of M on H) and τ (of N on K) we have $(E_1 \oplus E_2)^{\tau, \sigma} \cong E_1^{\tau, \sigma} \oplus E_2^{\tau, \sigma}$.*

Lemma 3.9 *Let E be a W^* -correspondence from M to N and F be a W^* -correspondence from N to Q . Let σ, τ and π be normal faithful representations of M, N and Q respectively. Then the map $X \otimes Y \mapsto (I_E \otimes X)Y$ (for $X \in F^{\pi, \tau}$ and $Y \in E^{\tau, \sigma}$) defines an isomorphism*

$$F^{\pi, \tau} \otimes E^{\tau, \sigma} \cong (E \otimes F)^{\pi, \sigma}.$$

Given a W^* -correspondence Z over M and two faithful representations σ and τ (of M), the dual correspondences Z^σ and Z^τ are, in general, non isomorphic but, as we shall show below they are Morita equivalent. We will also show that the converse holds. For this, we first recall the definition of Morita equivalence for W^* -correspondences.

In [13] we discussed Morita equivalence for C^* -correspondences. For W^* -correspondences the concept is defined similarly (with minor changes). Recall first that an M - N *equivalence bimodule* is an M - N bimodule X that is endowed with M - and N -valued inner products, ${}_M\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_N$, making X a full and selfdual (right) Hilbert W^* -module over N and a full and selfdual (left) Hilbert W^* -module over M such that ${}_M\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_N$ for ξ, η and ζ in X . By definition, the von Neumann algebras M and N are strongly Morita equivalent in case there is an M - N equivalence bimodule [26].

Definition 3.10 (cf. [13]) *W^* -correspondences E and F , over M and N respectively, are said to be (strongly) Morita equivalent if there is an M - N equivalence bimodule X such that $E \otimes_M X \cong X \otimes_N F$ (where the isomorphism here is a triple (id, Ψ, id) for some map Ψ).*

In the statement of the following theorem, the isomorphisms are in the sense indicated at the end of Remark 2.4.

Theorem 3.11 *Let E and F be W^* -correspondences over σ -finite von Neumann algebras M and N respectively. Then the following conditions are equivalent.*

- (1) *There is a W^* -correspondence Y (over some von Neumann algebra Q) and two faithful representations π_1 and π_2 of Q such that $E \cong Y^{\pi_1}$ and $F \cong Y^{\pi_2}$.*
- (2) *The W^* -correspondences E and F are strongly Morita equivalent.*
- (3) *There are faithful representations σ and τ , of M and N respectively, such that $E^\sigma \cong F^\tau$.*

Proof. We shall only sketch the proof. For a detailed proof one needs to be careful about the maps involved in each of the isomorphisms below (see Remark 2.4). To prove (3) implies (1), write ψ for the isomorphism from $\sigma(M)'$ onto $\tau(N)'$ (implied by the assumption that $E^\sigma \cong F^\tau$), write ι_1 and ι_2 for the identity representations of $\sigma(M)'$ and $\tau(N)'$ respectively and set $Y = E^\sigma$. Then it follows from duality that $E \cong Y^{\iota_1}$ and $F \cong Y^{\iota_2 \psi}$.

To prove (1) implies (2) let Z be the identity correspondence Q and set $X = Z^{\pi_1, \pi_2}$. Then $E \otimes X \cong Y^{\pi_1} \otimes Z^{\pi_1, \pi_2} \cong (Q \otimes Y)^{\pi_1, \pi_2} \cong (Y \otimes Q)^{\pi_1, \pi_2} \cong Z^{\pi_1, \pi_2} \otimes Y^{\pi_2} \cong X \otimes F$.

For the last part, assume E and F are Morita equivalent and X is the equivalence bimodule implementing the equivalence. Assume $N \subseteq B(K)$ and write H for $X \otimes_N K$. Write σ for the identity representation of N' (on K) and τ for the representation of N' on H defined by $\tau(a) = I_X \otimes a$. Note that

$\tau(N')' = \mathcal{L}(X) \otimes I_K$ and this algebra is isomorphic to M (since $M \cong \mathcal{L}(X)$ for an equivalence bimodule X). Write ψ for this isomorphism (from M to $\tau(N')'$). Let ι_1 be the identity representation of $\tau(N')'$ on H and ι_2 be the identity representation of N on K . Write Z for the identity correspondence of N' . Given $x \in X$, define $S(x)$ to be the map from K to $Z \otimes_\tau H$ given by $S(x)(k) = I \otimes_\tau (x \otimes k)$. It is easy to check that $S(x)$ lies in $Z^{\tau, \sigma}$. In fact, the triple (ψ, S, id) is an isomorphism of X and $Z^{\tau, \sigma}$. It then follows from duality that $F^{\iota_2} \cong F^{\iota_2} \otimes N' \cong F^{\iota_2} \otimes (Z^{\tau, \sigma})^{\iota_2, \iota_1} \cong (Z^{\tau, \sigma} \otimes F)^{\iota_2, \iota_1} \cong (X \otimes F)^{\iota_2, \iota_1 \psi} \cong (E \otimes X)^{\iota_2, \iota_1 \psi} \cong (E \otimes_\psi Z^{\tau, \sigma})^{\iota_2, \iota_1 \psi} \cong (Z^{\tau, \sigma})^{\iota_2, \iota_1} \otimes E^{\iota_1 \psi} \cong N' \otimes_\tau E^{\iota_1 \psi} \cong E^{\iota_1 \psi}$. \square

4 Applications of duality: Commutants and Canonical models

In this section, E will be a W^* -correspondence over a von Neumann algebra M . As was mentioned in the introduction, there are several applications for the dual correspondences. Here we concentrate on using the Hardy algebra associated with a dual correspondence in order to study the Hardy algebra associated with the original correspondence E . The first result along these lines is the identification of the commutant of $H^\infty(E)$ (given in some induced representation). The other is the development of canonical models to study representations of $H^\infty(E)$. As we shall see, the characteristic functions can be identified with elements of $H^\infty(E^\tau)$ for some dual correspondence E^τ .

Although $H^\infty(E)$ was defined as a subalgebra of $\mathcal{L}(\mathcal{F}(E))$ it is often useful to consider a (faithful) representation of it on a Hilbert space. Given a faithful, normal, representation σ of M on H we can “induce” it to a representation of the Hardy algebra. To do this, we form the Hilbert space $\mathcal{F}(E) \otimes_\sigma H$ and write

$$Ind(\sigma)(X) = X \otimes I, \quad X \in H^\infty(E).$$

(Note that $Ind(\sigma)(X)$ is a well defined bounded operator on $\mathcal{F}(E) \otimes_\sigma H$ for every X in $\mathcal{L}(\mathcal{F}(E))$. Such representations were studied by M. Rieffel in [27]). $Ind(\sigma)$ is a faithful representation and is a homeomorphism with respect to the σ -weak topologies. Similarly one defines $Ind(\iota)$ (where ι is the identity representation of $\sigma(M)'$ on H), a representation of $H^\infty(E^\sigma)$. The following theorem shows that, roughly speaking, the algebras $H^\infty(E)$ and $H^\infty(E^\sigma)$ are the commutant of each other. For the proof, see [16, Theorem 3.9].

Theorem 4.1 [16] *Let E be a W^* -correspondence over M and σ be a faithful normal representation of M on H . Then there exists a unitary operator $U : \mathcal{F}(E^\sigma) \otimes H \rightarrow \mathcal{F}(E) \otimes H$ such that*

$$U^*(Ind(\iota)(H^\infty(E^\sigma)))U = (Ind(\sigma)(H^\infty(E)))'$$

and, consequently,

$$(Ind(\sigma)(H^\infty(E)))'' = Ind(\sigma)(H^\infty(E)).$$

Example 4.2 Given an $n \times n$ matrix C in $M_n(\mathbb{Z}_+)$. One can associate with it a W^* -correspondence $E(C)$ over the algebra D_n of all diagonal $n \times n$ matrices. (See [12] for details). The tensor algebra associated with $E(C)$ is called the quiver algebra or the path algebra (associated with the directed graph defined by C). It is a subalgebra of the Cuntz-Krieger C^* -algebra O_C . If σ is the identity representation of D_n (on \mathbb{C}^n), then $E(C)^\sigma = E(C^t)$ (where C^t is the transpose matrix). Thus, Theorem 4.1 gives another proof of [12, Proposition 5.4] and of [7, Theorem 5.8]. (Note that a semigroupoid algebra of [7] is the image, under an induced representation, of a quiver algebra.)

Another way in which the Hardy algebra of the dual correspondence plays a role in studying $H^\infty(E)$ is through canonical models. Here, roughly, the elements of the Hardy algebra of the dual play the role that “Schur multipliers” play in the classical theory.

The canonical models are used to study certain representations of the Hardy algebra. The representations of $H^\infty(E)$ that we shall study here are the completely noncoisometric (abbreviated c.n.c) and the C_0 -representations. We first recall the definitions. (See [16] for more details).

Recall that, given a covariant representation (T, σ) of E on H , it has a unique minimal isometric dilation (V, π) on K . It was described in Section 2 and we use here the notation introduced there.

Definition 4.3 (1) A completely contractive covariant representation (T, σ) of E is called a C_0 -representation if, for every $h \in H$, $\|\tilde{T}_n^* h\| \rightarrow 0$. Equivalently, if $L_\infty(\mathcal{Q}_0) = K$.

(2) A completely contractive covariant representation (T, σ) of E is said to be completely noncoisometric if $L_\infty(\mathcal{D}) \vee L_\infty(\mathcal{Q}_0) = K$.

Clearly, every C_0 -representation is completely non coisometric.

Theorem 4.4 [16] If (T, σ) is a c.n.c representation then $T \times \sigma$ extends to a w^* -continuous representation of $H^\infty(E)$.

We now define characteristic operator functions in this context. It generalizes the classical case [19] and the case studied by Popescu [24]. Here we shall present the constructions and the main results. Full details will appear in [17].

Definition 4.5 Given a von Neumann algebra M and a W^* -correspondence E over M , A characteristic operator function is a tuple $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ such that

(i) For $i = 1, 2$, \mathcal{E}_i is a Hilbert space and τ_i is a representation of M on \mathcal{E}_i .

(ii) $\Theta : \mathcal{F}(E) \otimes_{\tau_1} \mathcal{E}_1 \rightarrow \mathcal{F}(E) \otimes_{\tau_2} \mathcal{E}_2$ is a contraction satisfying

$$(\varphi_\infty(a) \otimes I_{\mathcal{E}_2})\Theta = \Theta(\varphi_\infty(a) \otimes I_{\mathcal{E}_1}), \quad a \in M \quad (4)$$

and

$$(T_\xi \otimes I_{\mathcal{E}_2})\Theta = \Theta(T_\xi \otimes I_{\mathcal{E}_1}). \quad (5)$$

(iii) There is no non zero vector $x \in \mathcal{E}_1$ such that $x = P_{\mathcal{E}_1} \Theta^* P_{\mathcal{E}_2} \Theta x$. (We say that Θ is purely contractive).

(iv) $\overline{\Delta_{\Theta}(\mathcal{F}(E) \otimes \mathcal{E}_1)} = \overline{\Delta_{\Theta}((\mathcal{F}(E) \otimes \mathcal{E}_1) \ominus \mathcal{E}_1)}$ (where $\Delta_{\Theta} = (I - \Theta^* \Theta)^{1/2}$).

If, in addition, Θ is an isometry then it will be called an inner characteristic operator function. (In this case, (iv) holds automatically).

We shall often refer to Θ as the characteristic operator function (when \mathcal{E}_i and τ_i are assumed to be known).

Note that it follows from (4) and (5) that, if we write \mathcal{E} for $\mathcal{E}_1 \oplus \mathcal{E}_2$ and let τ be the representation $\tau_1 \oplus \tau_2$, then the matrix $\begin{pmatrix} 0 & 0 \\ \Theta & 0 \end{pmatrix}$ (viewed as an element of $\mathcal{F}(E) \otimes \mathcal{E}$) lies in the commutant of $\text{Ind}(\tau)(H^{\infty}(E))$. If τ is faithful we will be able to use Theorem 4.1 to conclude that the matrix defines an element of $H^{\infty}(E^{\tau})$. If τ is not faithful, one can form $\tau' = \tau \oplus \tau_0$ that is a faithful representation (on a larger space) and consider the 3×3 matrix with Θ in the 2, 1 entry instead of the 2×2 matrix above. Since this is just a technical point, we shall ignore it here and use Theorem 4.1 to define $\hat{\Theta} \in H^{\infty}(E^{\tau})$ by

$$\text{Ind}(\iota)(\hat{\Theta}) = U \begin{pmatrix} 0 & 0 \\ \Theta & 0 \end{pmatrix} U^*.$$

The left hand side will also be written $\hat{\Theta} \otimes I_{\mathcal{E}}$. Note that, if Θ is inner, $\hat{\Theta}^* \hat{\Theta}$ is the projection onto \mathcal{E}_1 .

Next we construct, for a fixed characteristic operator function Θ a covariant representation associated to it.

For this, we write $\Delta_{\Theta} = (I_{\mathcal{F}(E) \otimes \mathcal{E}_1} - \Theta^* \Theta)^{1/2} \in B(\mathcal{F}(E) \otimes \mathcal{E}_1)$ and set

$$K(\Theta) = (\mathcal{F}(E) \otimes \mathcal{E}_2) \oplus \overline{\Delta_{\Theta}(\mathcal{F}(E) \otimes \mathcal{E}_1)} \subseteq \mathcal{F}(E) \otimes \mathcal{E}$$

and

$$H(\Theta) = ((\mathcal{F}(E) \otimes \mathcal{E}_2) \oplus \overline{\Delta_{\Theta}(\mathcal{F}(E) \otimes \mathcal{E}_1)}) \ominus \{\Theta x \oplus \Delta_{\Theta} x : x \in \mathcal{F}(E) \otimes \mathcal{E}_1\}.$$

Note that, if Θ is inner, we get $\Theta^* \Theta = U^*(q_1 \otimes I_{\mathcal{E}})U = I_{\mathcal{F}(E)} \otimes q_1$ and $\Delta_{\Theta} = 0$. Thus, in this case, $K(\Theta) = \mathcal{F}(E) \otimes \mathcal{E}_2$ and $H(\Theta) = (\mathcal{F}(E) \otimes \mathcal{E}_2) \ominus \Theta(\mathcal{F}(E) \otimes \mathcal{E}_1)$.

We shall also write P_{Θ} for the projection from $K(\Theta)$ onto $H(\Theta)$. We have the following.

Theorem 4.6 *Let Θ be a characteristic operator function and let $K(\Theta)$ and $H(\Theta)$ be as above. For every $a \in M$ and $\xi \in E$ we define the operators $S_{\Theta}(\xi)$ and $\psi_{\Theta}(a)$ on $\Delta_{\Theta}(\mathcal{F}(E) \otimes \mathcal{E}_1)$ by*

$$S_{\Theta}(\xi) \Delta_{\Theta} g = \Delta_{\Theta} (T_{\xi} \otimes I_{\mathcal{E}_1}) g, \quad g \in \mathcal{F}(E) \otimes \mathcal{E}_1$$

and

$$\psi_{\Theta}(a) \Delta_{\Theta} g = \Delta_{\Theta} (\varphi_{\infty}(a) \otimes I_{\mathcal{E}_1}) g, \quad g \in \mathcal{F}(E) \otimes \mathcal{E}_1.$$

Also, we define on $K(\Theta)$ the operators

$$V_\Theta(\xi) = (T_\xi \otimes I_{\mathcal{E}_2}) \oplus S_\Theta(\xi)$$

and

$$\rho_\Theta(a) = (\varphi_\infty(a) \otimes I_{\mathcal{E}_2}) \oplus \psi_\Theta(a).$$

Then

- (i) (S_Θ, ψ_Θ) and (V_Θ, ρ_Θ) define isometric covariant representations of E on $\Delta_\Theta(\mathcal{F}(E) \otimes \mathcal{E}_1)$ and $K(\Theta)$ respectively.
- (ii) The space $K(\Theta) \ominus H(\Theta)$ is invariant for (V_Θ, ρ_Θ) and, thus, the compression of (V_Θ, ρ_Θ) to $H(\Theta)$, which we denote by $(T_\Theta, \sigma_\Theta)$, is a completely contractive covariant representation of E . Explicitly,

$$T_\Theta(\xi) = P_\Theta V_\Theta(\xi)|_{H(\Theta)}, \quad \xi \in E$$

and

$$\sigma_\Theta(a) = P_\Theta \rho_\Theta(a)|_{H(\Theta)}, \quad a \in M.$$

- (iii) The representation $(T_\Theta, \sigma_\Theta)$ is completely non coisometric. It is a $C_{0,0}$ -representation if and only if Θ is inner.

The converse of the theorem above also holds; that is, every c.n.c representation of E gives rise to a characteristic operator function. To construct it we now fix such a representation and let $V, \pi, K, K_0, \mathcal{D}$ and \mathcal{D}_* be as in Section 2. Also, let ρ_1 be the restriction of π to \mathcal{D} and ρ_2 be the restriction of π (or σ) to \mathcal{D}_* . (Again, we shall assume that $\rho = \rho_1 \oplus \rho_2$ is faithful. Otherwise a minor technical correction is needed). We also write $G = \mathcal{D} \oplus \mathcal{D}_*$.

Associated with the wandering subspaces \mathcal{Q}_0 and \mathcal{D} of K we have the unitary operators $W(\mathcal{Q}_0)$ and $W(\mathcal{D})$ defined above (see the discussion preceding Theorem 2.14). We write Q_∞ for the projection of K onto $L_\infty(\mathcal{Q}_0)$ (=the range of $W(\mathcal{Q}_0)$).

Also recall that u is an isometry from \mathcal{Q}_0 onto \mathcal{D}_* that commutes with $\pi(a)$ for $a \in M$. It induces an isometry, written $I_{\mathcal{F}(E)} \otimes u$ from $\mathcal{F}(E) \otimes \mathcal{Q}_0$ onto $\mathcal{F}(E) \otimes \mathcal{D}_*$.

We now write

$$\Theta_T = (I_{\mathcal{F}(E)} \otimes u)W(\mathcal{Q}_0)^*Q_\infty W(\mathcal{D}) : \mathcal{F}(E) \otimes \mathcal{D} \rightarrow \mathcal{F}(E) \otimes \mathcal{D}_*. \quad (6)$$

Theorem 4.7 *Given a c.n.c representation (T, σ) of E on a Hilbert space H , the tuple $(\Theta_T, \mathcal{D}, \mathcal{D}_*, \rho_1, \rho_2)$, defined above, is a characteristic operator function in the sense of Definition 4.5. The representation is a $C_{0,0}$ -representation if and only if the characteristic operator function is inner.*

Moreover, the representation constructed from Θ_T as in Theorem 4.6 is unitarily equivalent to (T, σ) .

We also have the following.

Theorem 4.8 *Suppose we start with a characteristic operator function $(\Theta, \mathcal{E}_1, \mathcal{E}_2, \tau_1, \tau_2)$ and write $(T_\Theta, \sigma_\Theta)$ for the c.n.c representation constructed in Theorem 4.6. With this representation we can associate the characteristic operator function $(\Theta_T, \mathcal{D}, \mathcal{D}_*, \rho_1, \rho_2)$ as in Theorem 4.7.*

Then, the two characteristic operator functions are unitarily equivalent in the sense that there are unitary operators $W_1 : \mathcal{E}_1 \rightarrow \mathcal{D}$ (intertwining τ_1 and ρ_1) and $W_2 : \mathcal{E}_2 \rightarrow \mathcal{D}_$ (intertwining τ_2 and ρ_2) such that*

$$\Theta_T = (I_{\mathcal{F}(E)} \otimes W_2)\Theta(I_{\mathcal{F}(E)} \otimes W_1^*).$$

The above results show that the characteristic operator functions (or the elements $\hat{\Theta}$ obtained from these) serve as complete invariants for c.n.c representations of $H^\infty(E)$.

We also state the following result which we know only for $C_{0,0}$ -representations. In this result we refer to compositions $\Theta = \Theta_1\Theta_2$ where Θ is the inner characteristic operator function associated with a $C_{0,0}$ -representation and Θ_i , $i = 1, 2$, is an inner characteristic operator function but is not necessarily purely contractive. Two such compositions $\Theta = \Theta_1\Theta_2 = \Theta'_1\Theta'_2$ are said to be *equivalent* if $\Theta'_1 = \Theta_1(I \otimes V_0)$ (and $\Theta'_2 = (I \otimes V_0^*)\Theta_2$) for some unitary operator V_0 .

Theorem 4.9 *Let (T, σ) a $C_{0,0}$ -representation of E on H (with $\sigma \times T$ the associated representation of $H^\infty(E)$). Let Θ be the inner characteristic operator function of this representation. Then there is a bijection between the subspaces of H that are $H^\infty(E)$ -invariant and (equivalence classes of) factorizations $\Theta = \Theta_1\Theta_2$ of Θ as a composition of two inner characteristic operator functions (that are not necessarily purely contractive).*

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